AN AXIOMATIC BASIS FOR PLANE GEOMETRY*

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1. **The axioms.** The fourth appendix of Hilbert's *Grundlagen der Geometrie*† is devoted to the foundation of plane geometry on three axioms pertaining to transformations of the plane into itself. The object of the present paper is to attain the same end by quicker and simpler means. The simplifications are made possible by using orientation-reversing transformations‡ and changing Hilbert's second and third axioms.

The (x, y)-plane will mean the set of all distinct ordered pairs of real numbers. The terms of analytic geometry, to which no geometric content need be given, will be used, modified by the prefix (x, y) where ambiguity might arise. Thus we shall refer to (x, y)-*distance*, (x, y)-*lines*, and so on.

The *general plane*, p, will be any set of objects, called *points*, which can be put in one-to-one correspondence with the points of the (x, y)-plane. For convenience, we shall speak of the points of p as if they were identical with their images under such a correspondence.

The following axioms pertain to a set, T, of continuous§ one-to-one transformations of p into itself. A transformation of the set which leaves two distinct points fixed and reverses orientation will be called a *reflection*.

Axiom 1. *The transformations T form a group.*

Axiom 2. *If A and B are two points of p, T contains a reflection leaving A and B fixed.*

Axiom 3. *Let T_A denote the subset of T containing all the transformations thereof which leave A fixed. If T_A contains transformations carrying pairs of points arbitrarily near a given pair of points (B, C) into an arbitrarily small neighborhood of a pair (D, E), then T_A contains a transformation carrying (B, C) into (D, E).*

2. **The curve γ.** Our first object is to establish the following theorem, which, like Lemma 1 below, is similar to a result employed by Hilbert (loc. cit.).

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† D. Hilbert, *Grundlagen der Geometrie*, 1930, pp. 178–230.

‡ Suggested by Hilbert, loc. cit., p. 182.

§ That is, continuous in terms of (x, y)-distance.

THEOREM 1. *Every neighborhood of A contains a simple closed curve, γ, enclosing A and preserved by each of the transformations T_A.*

We shall assume the Jordan separation theorem and the following converse thereof:

(A) *A locally connected* set of points which divides the (x, y)-plane into two regions, one of them finite, and forms their common boundary, is a simple closed curve.*†

LEMMA 1. *For any given positive ϵ, there is a neighborhood, N, of A on p, no point of which is carried to a distance ϵ from A by any of the transformations T_A (Axiom 3).*

Otherwise, let P_i $(i=1, 2, \cdots)$ be a point within distance $\epsilon/2^i$ of A, whose image, Q_i, under one of the transformations T_A is at distance ϵ from A. Then T_A contains transformations carrying points arbitrarily near A into an arbitrarily small neighborhood of any cluster point, Q, of $(Q_1, Q_2, \cdots)$. Therefore, by Axiom 3, T_A contains a transformation carrying A into Q. But this is impossible, for the transformations T_A all leave A fixed.

LEMMA 2. *Let c be any simple closed curve on N enclosing A. The set, Γ, of all points into which points on c are carried by the transformations T_A is closed. The transformations T_A all preserve Γ.*

Any cluster point, P, of Γ is limit of some series $(P_1, P_2, \cdots)$ on Γ. By definition of Γ, one of the transformations T_A carries a point Q_i $(i=1, 2, \cdots)$ on c into P_i. Hence, if Q is a cluster point, necessarily on c, of $(Q_1, Q_2, \cdots)$, T_A (see Axiom 3) contains transformations carrying points arbitrarily near Q into an arbitrary neighborhood of P. Therefore (Axiom 3), T_A contains a transformation carrying Q into P. Hence P is on Γ, and Γ is closed.

Consider the image, P', of any point, P, on Γ under any transformation, T_1, of the set T_A. Let T_0 be one of the transformations T_A carrying some point, Q, on c into P. Then T_0T_1 carries Q into P'. Since T_0T_1 leaves A fixed and belongs to T (Axiom 1), it belongs also to T_A. Therefore P' is on Γ. This completes the proof.

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* A point set, S, is said to be *locally connected* if, for any $\epsilon>0$ and any point P, of S, there exists a positive distance, δ, such that all points of S within distance δ of P are connected with P by a subset of S entirely within distance ϵ of P.

† Essentially in this form, the theorem is given by J. R. Kline, these Transactions, vol. 21 (1920), p. 452. It is a ready consequence of Hahn's characterisation of continuous curves, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 23 (1914), p. 318, together with a theorem by R. L. Moore, Bulletin of the American Mathematical Society, vol. 23 (1917), p. 233, that any two points of a continuous curve, S, can be joined on S by a simple Jordan arc.

LEMMA 3. *The complement of* Γ *on the* (x, y)*-plane contains just one unlimited region,* R. *The boundary,* γ, *of* R *divides the* (x, y)*-plane into two regions,* R *and* R_0, *and forms their common boundary.*

The first part of the lemma follows from Lemmas 1 and 2. It also follows from these lemmas that γ is on Γ. Let P denote any point neither in R nor on γ, and k a simple arc through P with just its end points, P_1 and P_2, on γ. Some transformation, T_j $(j=1, 2)$, of the set T_A carries a point, Q_j, of c into P_j (Lemma 2). A simple arc inside c joining A to Q_j is carried by T_j into an arc, k_j, joining A to P_j but not meeting either R or γ. Because R is connected, $(k+k_1+k_2)$ cannot enclose any point of either R or its boundary, γ. Therefore γ cannot separate P from A. Hence all points neither in R nor on γ are in a single region, R_0. By such a curve as k_1, any point on γ can be joined to A inside R_0. Therefore, γ is the common boundary of R_0 and R_1.

LEMMA 4. *The boundary* γ *is locally connected.*

Suppose that at some point, X, γ is not locally connected. Then a positive number, d, exists, so small that every neighborhood of X contains points on γ not connected with X by any continuum on γ entirely within distance d of X. Let P_i $(i=1, 2, \cdots)$ be one such point within distance $d/2^i$ of X. Let C denote the (x, y)-circle of radius d about X and K_i the set of all points connected with P_i on γ inside C. Then, if K_i and K_j have a point in common, they coincide. It may readily be seen that K_i contains all its cluster points inside C. Therefore, at most a finite number of the K's can coincide with any one of them. Otherwise, an infinite subset of $(P_1, P_2, \cdots)$ would belong to one of the K's, which would therefore contain X and join it in C to certain of the P's. Hence, with no loss of generality, we may assume that the K's are all distinct.*

Let C_1 be the circle of radius $d/2$ about X, and K_i' $(i=1, 2, \cdots)$ a closed connected subset of K_i joining P_i to C_1 but containing no points outside C_1. Let C_2 be the circle of radius $d/4$ about X. Without loss of generality, we assume* that K_i' contains a point, Q_i, on C_1 and a point, S_i, on C_2 such that $(Q_1, Q_2, \cdots)$ converges to a limit, Q, monotonically on the arc Q_1Q_2Q, and $(S_1, S_2, \cdots)$ converges to a limit, S, monotonically on the arc S_1S_2S. Consider, for any $i>1$, a simple closed curve made up of two arcs, k_1 and k_2, joining A to S_i inside R_0 (Lemma 3, proof). Suppose k_1 meets the arc $\alpha_i \equiv Q_{i-1}Q_iQ_{i+1}$ on C_1, but not the broken line $\beta_i \equiv P_{i-1}P_iP_{i+1}$, whereas k_2 meets β_i but not α_i. Then (k_1+k_2) separates S_{i-1} from S_{i+1}. For, let a simple

* To avoid excessive notation, we assume for the K's several properties enjoyed by some infinite subset thereof.

closed curve, γ_0, be formed by adding to α_i and β_i a pair of arcs joining Q_{i-1} to P_{i-1} and Q_{i+1} to P_{i+1}, respectively. Let these latter curves pass through S_{i-1} and S_{i+1}, respectively, and lie so near to K'_{i-1} and K'_{i+1} that γ_0 encloses S_i and is met by k_1 only on α_i and by k_2 only on β_i. Then (k_1+k_2) clearly contains just one arc inside γ_0 separating S_{i-1} from S_{i+1}. Therefore S_{i-1} and S_{i+1} are separated by the closed curve (k_1+k_2). But this is impossible, for no curve in R_0 can enclose points of R (Lemma 3). Therefore, if c_i is a simple closed curve in R_0 through A and S_i, then α_i (or β_i) meets both of the arcs into which c_i is divided by A and S_i.

Now S_i $(i=1, 2, \cdots)$, being on Γ (Lemmas 2, 3), is image, under a transformation, T_i, of the set T_A, of some point, E_i, on the curve c of Lemma 2. Let c' be a simple closed curve through A which contains no points outside c but has in common with c an arc through a cluster point, E, of $(E_1, E_2, \cdots)$. With no loss of generality, we assume* that all the E's lie on c' and that $(E_1, E_2, \cdots)$ converges to E. Then T_i carries c' into a simple closed curve, c_i, to which the conclusion of the preceding paragraph applies.† We shall treat only the case where both the arcs AS_i $(i=1, 2, \cdots)$ on c_i meet α_i.‡ In this case, c' passes through two points, E_i' and E_i'', separated on c' by (A, E_i), where the images of (E_i', E_i'') under T_i are on α_i. Without loss of generality, we assume* that $(E_1', E_2', \cdots)$ and $(E_1'', E_2'', \cdots)$ converge to a pair of points, E' and E'', respectively. Now, by definition, α_i, for i large enough, is in an arbitrarily small given neighborhood of Q. Hence, since T_i carries (E_i', E_i'') onto α_i, T_A contains a transformation (Axiom 3) carrying (E', E'') into Q. Hence $E'\equiv E''$. Since A and E_i separate E_i' and E_i'' on c', E' and E'' can coincide only at A or at E. But A cannot go into Q under any of the transformations T_A. Hence $E'\equiv E''\equiv E$. Then, for i large enough, T_i carries a pair of points (E_i, E_i') arbitrarily near E into an arbitrary neighborhood of the pair (S, Q). Therefore (Axiom 3) T_A contains a transformation carrying E' into (S, Q). This contradicts the one-to-one-ness of the transformations and establishes the lemma.

Theorem 1 *above is an immediate consequence of* (A) *together with Lemmas* 2, 3, *and* 4.

3. **Lines and reflections.** The set of all fixed points under a reflection (see §1) will be called a *line*.

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* See footnote on p. 236.

† To show this, c' may be slightly deformed, if necessary, so that its image meets γ only at S_i.

‡ The method applies equally well if both arcs meet β_i. We need only replace α by β and Q by P. In assuming that the arc AS_i meets α_i (or β_i) for all values i, we employ the convention stated in the footnote on p. 236.

LEMMA 1. *An orientation-preserving transformation, τ, of the group T which preserves a simple closed curve, γ, and leaves one point of γ fixed, leaves every point of γ fixed.**

If A denote the known fixed point on γ, then τ and all its powers belong to the set T_A (Axiom 3). Let P_i be the image of P_0 under the ith power of τ. Ascribing a positive sense to γ, consider the arc $\overrightarrow{AP_0}$. If it passes through P_1, then the arc $\overrightarrow{AP_1}$, being the image under τ of $\overrightarrow{AP_0}$, passes through P_2; and, in general, $\overrightarrow{AP_i}$ $(i=2, 3, \cdots)$ passes through P_{i+1}, but not P_{i-1}. On the other hand, if $\overrightarrow{P_0A}$ on γ passes through P_1, then $\overrightarrow{P_iA}$ contains P_{i+1} but not P_{i-1}. Thus, in either case, $(P_1, P_2, \cdots)$ is a monotonic series on the curve γ. If P is its limit, then, for i sufficiently large, the two points (P_i, P_{i+1}) are in an arbitrary preassigned neighborhood of P. But these two points are images, under τ^i, of (P_0, P_1). Therefore (Axiom 3) some transformation of the set T_A carries both P_0 and P_1 into P. This implies that P_0 and P_1 coincide, and hence that every point of C is fixed under τ.

COROLLARY. *The transformation τ is the identity.*

First, since τ is continuous, the set, S, of its fixed points is closed with respect to p. Suppose S does not coincide with p, and consider the largest connected subset, S_1, of S which contains C. Let k be a simple curve joining a point, Q, of $(p-S)$ to a point, B, of S_1, but not containing any point of (S_1-B). Let γ' be a simple closed curve about B which meets both k and S_1 and which is carried into itself by τ (Theorem 1). Since γ' meets S_1, all its points are fixed under τ and hence belong to S_1. This contradicts the definition of k and thus establishes the corollary.

LEMMA 2. *A simple closed curve which is preserved by a reflection, ρ, is met in just two points by the line which ρ defines.*

For, an orientation-reversing transformation which preserves a simple closed curve leaves just two points on the curve fixed.

LEMMA 3. *The identity is the only orientation-preserving transformation of the group T which leaves every point of a line, L, fixed.*

This follows from the preceding results of this section applied to the curve γ of Theorem 1, where A is on L.

THEOREM 2. *A reflection, ρ, is involutory. No two different reflections define the same line, L.*

* The proof is patterned after one by Hilbert, loc. cit., pp. 204, 205.

By Lemma 3, $\rho\rho$ is the identity. Also, if ρ and ρ' both define L, $\rho\rho'$ is the identity. Therefore $\rho=\rho'$.

COROLLARY. *A reflection preserves the set of all lines.*

Let L_1 be any line and L_2 its image under an arbitrary reflection, ρ. Let ρ_1 be the reflection defining L_1. Then, since reflections are involutory, $\rho\rho_1\rho$ leaves just the points on L_2 fixed and reverses orientation. Therefore $\rho\rho_1\rho$ is a reflection and L_2 a line.

4. **Properties of the line.** (A) *Let ρ be a reflection and (P_1, P_2) a pair of points interchanged by ρ (Theorem 2). Then any simple Jordan arc, k, which joins P_1 and P_2 meets the line, L, defined by ρ.*

Since ρ is involutory (Theorem 2), it interchanges k with its image, k'. As a point, Q, traces k from P_1 to P_2, its image, Q', under ρ traces k' from P_2 to P_1. As the arc P_1Q on k increases, Q reaches a first position, Q_1, which is on the image, under ρ, of the arc P_1Q, end points included. Let Q_2 be the image of Q_1. If Q_1 is not on L, the arcs Q_1Q_2 on k and k', respectively, have only their end points in common. Since ρ interchanges these arcs and reverses orientation, it must leave Q_1 and Q_2 fixed. Therefore $Q_1(\equiv Q_2)$ is on L.

LEMMA. *A line, L, is locally connected.*

Under a contrary assumption, let A denote a point at which L is not locally connected; and let c denote an (x, y)-circle about A, so small that every neighborhood of A contains points of L not connected with A on L inside c. In particular, consider the neighborhood N of Lemma 1 in §2, where ϵ is the radius of c. Let k be a simple arc in N, with only its end points on L, joining two points not connected on L inside c. Then, by (A), k and its image, k', under the reflection defining L are distinct and are separated by the points of L inside $(k+k')$. Hence these points of L join the end points of k; but this is contradictory, since, by definition of N, $(k+k')$ is entirely inside c.

(B) *If c is a simple arc with just its end points on a given line, L, then c and its image, γ, under the reflection, ρ, defining L, form a simple closed curve. The points of L inside $(c+\gamma)$ constitute a simple Jordan arc joining the common end points of c and γ.*

From (A), $(c+\gamma)$ is a simple closed curve. Let R be the set of all points which can be connected with c by a simple arc inside $(c+\gamma)$ containing no points of L. Let λ denote the set of points common to L and the boundary of R. The image, R', of R under ρ obviously consists of all the points which can be connected with γ inside $(c+\gamma)$ without meeting L. By (A), R and R' are distinct. Therefore λ separates c from γ and connects their common end points. Also, $(R+\lambda+R')$ contains all points inside $(c+\gamma)$, for otherwise some point would belong to a finite region bounded solely by points on L, and this region would go into itself under ρ in contradiction with (A). Now $(c+\lambda)$ divides the (x, y)-plane into two regions, the finite region R and the region consisting of R' plus γ plus the exterior of $(c+\gamma)$; and $(c+\lambda)$ is the common boundary of these two regions. Hence, using the above Lemma and §2(A), $(c+\lambda)$ is a simple closed curve, and λ is an arc thereof.

(C) *A line, L, is homeomorphic with the x-axis.*

Let c_1 be a simple arc with just its end points (A_1, B_1) on L, and let Q be a point of c_1. Then c_1 plus its image, γ_1, under the reflection, ρ, defining L cuts from L a Jordan arc, λ_1 (see (B)). Let λ_1 be put in continuous one-to-one correspondence with the segment $-1 \leqq x \leqq 1$ on the x-axis, A_1 corresponding to -1 and B_1 to $+1$.

Proceeding inductively, for $i=1, 2, \cdots$, let c_{i+1} be a simple arc which passes through Q, has only its end points on L, and lies outside $(c_i+\gamma_i)$. Further, let c_{i+1} pass through no point within some positive preassigned distance, d, of the points A_i and B_i. The curve c_{i+1} plus its image, γ_{i+1}, under ρ cuts from L a simple arc, λ_{i+1}, containing λ_i. Let λ_{i+1} be put in continuous one-to-one correspondence with the interval $-(i+1) \leqq x \leqq (i+1)$ on the x-axis in such a way as to preserve the correspondence of λ_i with the interval $-i \leqq x \leqq i$. Let (A_{i+1}, B_{i+1}) be the end points of λ_{i+1} which correspond to $-(i+1)$ and $(i+1)$, respectively. By the last condition imposed on c_{i+1}, neither of the series $(A_1, A_2, \cdots)$ and $(B_1, B_2, \cdots)$ has a cluster point on p. Hence the above inductive process leads to a continuous one-to-one correspondence between the x-axis and a portion, L', of L where L' divides p into two parts. Now the set of all points each inside at least one of the curves $(c_i+\gamma_i)$ $(i=1, 2, \cdots)$ is a neighborhood of L' free from other points of L. Hence L' and $(L-L')$ are distinct. If the set $(L-L')$ is not vacuous, let c' be an arc with just its end points on L, one end point being on L' and one on $(L-L')$. By (B), the end points of c' are joined on L by a simple arc. This contradiction establishes that $L' \equiv L$.

(D) *If two lines, L and L', through any point, A, have any other point, B, of the neighborhood N* (§2, *Lemma* 1) *in common, they coincide.*

Let γ_0 be a curve about A satisfying Theorem 1 and not enclosing B. Suppose γ_0 passes through no common point of L and L'. As the arc AB on L is traced from A, let B' be the first point reached outside γ_0 and on L'. Then B' is joined to γ_0 by two distinct arcs, k and k', on L and L', respectively. Let c be the simple closed curve about A formed of k, k' and an arc on γ_0. Using this for the curve c of Lemmas 2, 3, etc., in §2, we are led to a curve, γ, satisfying Theorem 1. This curve passes through B'. Suppose it does not. Then A and B' are both inside γ; and, since L and L' meet γ each in just two points, the arcs AB' on L and L' respectively, are both inside γ. Hence c and γ meet only on γ_0. But then* $\gamma=\gamma_0$, which is impossible. Since, therefore, γ passes through the common point B' of L and L', the product of the reflections defining L and L' is the identity (§3, Lemma 1 and Corollary), and $L\equiv L'$ (Theorem 2).

5. **Further properties. The connection with euclidean plane geometry.** The remaining developments prepare for a complete deduction of euclidean plane geometry.

LEMMA 1. *Let* (L_0, L_1) *be two different lines through any point,* A, *and let* (A_j, B_j) $(j=0, 1)$ *be the two points* (§3, *Lemma* 2) *in which* L_j *meets the curve* γ *of Theorem* 1. *Then the points* (A_0, B_0) *separate* A_1 *from* B_1 *on* γ.

Suppose the contrary, and let γ' denote the arc A_0B_0 on γ containing A_1 and B_1. Under the reflection defining L_1, L_0 goes into a line, L_2, which meets γ in the images (A_2, B_2) of (A_0, B_0), and, by §4(A), both A_2 and B_2 must be on the arc A_1B_1 of γ'. Proceeding inductively for $i=1, 2, \cdots$, let L_{2i+1} and L_{2i+2} be the images of L_1 and L_0, respectively, under the reflection, ρ_{2i}, defining L_{2i}. Let (A_{2i+1}, A_{2i+2}) be the images under ρ_{2i} of (A_1, A_0) and (B_{2i+1}, B_{2i+2}) the images of (B_1, B_0). Then the arc A_jB_j on γ' contains A_{j+1} and B_{j+1}. Thus the series $(A_1, A_2, \cdots)$ converges monotonically on γ' to a limit, X. Hence, for i large enough, A_{2i+1} and A_{2i+2} are within an arbitrary given distance of X. By Axiom 3, since ρ_{2i} belongs to T_A, T_A contains a transformation carrying (A_0, A_1) into X. But our transformations are all one-to-one. This contradiction establishes the desired result.

Let (A, P) denote any pair of distinct points on the plane p. The set of all images of P under reflections leaving A fixed will be called a *circle,* with A as *center.*

THEOREM 3. *A circle, K, is a simple closed curve.*

(I) Suppose the point P of the above definition is in a neighborhood, N, of A satisfying Lemma 1 in §2, so that K is in a finite region of the (x, y)-plane. Each point, P', on K is image of P under just one reflection leaving A fixed. For suppose there were two such reflections (ρ_1, ρ_2) carrying P into P'. Let ρ be the reflection defining the line AP'. Then $\rho_1\rho\rho_2$ and $\rho_1\rho\rho_1$ are both reflections leaving A and P fixed. Hence they both define the unique line (§4(D)) through A and P. Therefore $\rho_1\rho\rho_2\equiv\rho_1\rho\rho_1$ (Theorem 2), or $\rho_1\equiv\rho_2$.

Using the notation of Lemma 1 above, let γ' be one of the two arcs A_0B_0 on γ. As a point, Q, traces γ' from A_0 to B_0, the line through A and Q adopts

* Since γ is the set of images under T_A of points common to c and γ (Theorem 1, and Lemmas 2, 3 in §2), and T_A preserves γ_0.

the position of every line through A once and only once (Lemma 1), except that the same line is obtained for $Q \equiv A_0$ as for $Q \equiv B_0$. To eliminate the exception, regard A_0 and B_0 as identical, so that Q traces essentially a simple closed curve. Then the image, P', of P under the reflection defining the line AQ adopts every position on K once and only once as Q traces γ'. This affords a one-to-one correspondence between the points on K and those on γ', the point $A_0(\equiv B_0)$ included. It remains only to show the continuity of this correspondence. Let Q be any point on γ' and $(Q_1, Q_2, \cdots)$ a series of points converging to Q. Let P_i $(i=1, 2, \cdots)$ be the point on K corresponding to Q_i and let P^0 be any cluster point of the P's. Then there are transformations leaving A fixed, carrying points arbitrarily near Q into themselves and carrying P into an arbitrary neighborhood of P^0. Hence (Axiom 3) there is a transformation leaving A and Q fixed and carrying P into P^0. This must be the reflection in the line AQ (§4 (D); §3, Lemma 3 and Theorem 2). Therefore P^0 is the point corresponding to Q. This completes the argument for this special case.

(II) Suppose the theorem false. Then, by (I), as some line, L, is traced from A in one sense or the other, a point, P, is reached which is either the last position for which K is a simple closed curve, or the first for which it is not. In the first case, by a proof like that of Lemma 1 in §2, K has a neighborhood consisting of points which remain within distance ϵ of K under all transformations of the set* T_A. Since this neighborhood contains P it follows from (I) that P cannot be the last point for which K is a simple closed curve.

We deal with the second case by showing that if every internal point of the arc AP on L generates a circle which is a simple closed curve, then the circle, K, generated by P is also a simple closed curve. Let it first be noted that no two different lines (L_1, L_2) through A can meet at a point, Q, on K. If they did, then, by §4(D), the arcs AQ on L_1 and L_2 would be distinct and hence form a simple closed curve, c. But then every line determined by A and a point inside c would clearly pass through Q. By reflections in L_1 and L_2 one can show that other lines through A pass through Q; indeed, one can show that all lines through A pass through Q, from which it is easy to deduce a contradiction. Now, for any $\epsilon>0$, there exists a neighborhood, N_ϵ, of P, such that no image of P under the reflection defining a line through a point of N_ϵ is at distance greater than ϵ from P. This may be established by an argument similar to that of Lemma 1 in §2. Consider the correspondence employed in (I) above. We have seen that it is one-to-one even in the present case. We can establish its continuity by an argument like that in (I) applied

* The proof of Corollary 1 below shows that these transformations preserve K.

to images of P in N_ϵ, and, similarly, in a neighborhood of any point on K. Hence K is a simple closed curve.

COROLLARY 1. *A circle with center at A is preserved by all the transformations T_A (Axiom 3).*

Let τ be a transformation of the set T_A. Let P'' be the image under τ of any point P' on K, and let ρ be the reflection which leaves A fixed and interchanges (P, P'). Then, if ρ' is the reflection defining the line AP', $\rho\tau$ and $\rho\rho'\tau$ both carry P into P'' and leave A fixed. The one which reverses orientation is a reflection, for it preserves the curve γ of Theorem 1 and therefore leaves two of its points fixed. Hence P'' is on K.

COROLLARY 2. *A circle is met in just two points by a line through its center.* (See §3, Lemma 2.)

COROLLARY 3. *Any two points (P_1, P_2) on a circle are interchanged by some reflection leaving the center fixed.*

Let ρ_j be the reflection leaving the center, A, fixed and carrying P into P_j $(j=1, 2)$. If ρ define the line through A and P, the product $\rho_1\rho\rho_2$ reverses orientation, carries P_1 into P_2 and preserves the circle. It is therefore a reflection, for it leaves two points on the circle fixed. Hence (Theorem 2) it interchanges P_1 and P_2.

THEOREM 4. *One and only one line passes through any two distinct points of the plane p.*

If two lines through a point, A, have a second point, P, in common, their defining reflections preserve the circle through P with center at A and leave P fixed. The product of these reflections is therefore the identity and the lines coincide (§3).

COROLLARY 1. *For any two points, A and B, on the plane p, there exists a reflection interchanging them.*

Let P be a common point of the two circles, centers at A and B, respectively, passing through B and A, respectively. Let ρ_1 be the reflection interchanging A and P and leaving B fixed (Theorem 3, Corollary 3) and ρ_2 the reflection interchanging B and P and leaving A fixed. Then $\rho_1\rho_2\rho_1$ reverses orientation, leaves P fixed, and carries A into B. Since ρ_1 and ρ_2 are involutory (Theorem 2), $\rho_1\rho_2\rho_1$ is also involutory and therefore leaves more than one point fixed (see proof of §4(A)). Hence it is the required reflection.

COROLLARY 2. *Let L and L' be any two lines. Let A be any point on L and B any point on L'. Then some transformation in the group T carries L into L' in such a way that A goes into B.*

The reflection, ρ_1, which interchanges A and B (Corollary 1) carries L into a line, L'', through B (Theorem 2, Corollary). Some reflection, ρ_2, by Theorem 3, Corollary 3, leaves B fixed and interchanges any two points in which L' and L'', respectively, meet a circle, center at B. The product $\rho_1\rho_2$ satisfies the requirements of the present corollary.

There remain no difficulties in defining angles, distances, congruences, and proceeding with other geometrical developments, or else establishing the axioms in Chapter 1 of Hilbert's *Grundlagen der Geometrie*.

Two geometries rest on the above foundation: the euclidean if one assume that through a given point not on a given line, L, there is but one line which fails to meet L; the Bolyai-Lobatchewsky if one assume that there are two lines through the point separating the intersecting from the non-intersecting lines.

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